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ON THE SUPERRADIANCE OF SPIN-1 WAVES IN AN EQUATORIAL WEDGE AROUND A KERR HOLE

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ABSTRACT

Recently Van Putten has suggested that superradiance of magnetosonic waves in a toroidal magnetosphere around a Kerr black hole may play a role in the central engine of γ-ray bursts. In this context, he computed (in the WKB approximation) the superradiant amplification of scalar waves confined to a thin equatorial wedge around a Kerr hole and found that the superradiance is higher than for radiation incident over all angles. This paper presents calculations of both spin-0 (scalar) superradiance (integrating the radial equation rather than using the WKB method) and and spin-1 (electromagnetic/magnetosonic) superradiance, in Van Putten's wedge geometry. In contrast to the scalar case, spin-1 superradiance decreases in the wedge geometry, decreasing the likelihood of its astrophysical importance.

Subject headings: black hole physics – gamma rays: bursts

1. INTRODUCTION

Van Putten (1999) has proposed that superradiant scattering of magnetosonic waves by a Kerr black hole plays an important role in the central engine of γ-ray bursts. In his model, the tidal breakup of a magnetized neutron star as it spirals into a Kerr hole creates a massive torus and a toroidal magnetosphere. Inside, there is a current-free cavity which acts as a waveguide for fast magnetosonic waves bouncing between the horizon and the torus. As shown by Uchida (1997), in the geometrical-optics limit these waves (though not the Alfven waves) obey the same equations as the vacuum electromagnetic waves (with 'spin-weight' $s = 1$) and hence are amplified by 'superradiant' scattering. As the waves are assumed to be perfectly confined to the cavity, this leads to an instability in all superradiant modes. To calculate the degree of superradiance, Van Putten approximates the geometry as the interior of a very thin equatorial wedge extending from the horizon to infinity (where the torus lies). This leads to a simple prescription for the angular eigenvalues in the separated wave equation, and these can then be used in the integration of the radial equation to obtain reflection and transmission coefficients. With this method Van Putten finds that scalar waves ($s = 0$) are reflected with a superradiance about ten times that calculated in the full spheroidal geometry (Van Putten 1999; Press & Teukolsky 1972). This is notable because (in the full geometry) superradiance increases with spin-weight, so this suggested that magnetosonic ($s = 1$) superradiance in the wedge may be more efficient than either scalar superradiance in the wedge or $s = 1$ superradiance over the full angular scale. In this letter I calculate the $s = 1$ superradiance using a method analogous to Van Putten's and find that the superradiance decreases, rather than increases, in the 'wedge' geometry.

2. METHOD OF SOLUTION

In the Newman-Penrose (1962) formalism, all field quantities are represented by potentials obtained by projecting the fields onto a complex tetrad of null-vectors $(l^\nu, n^\nu, m^\nu, m^{*\nu})$ which satisfy $l_\nu n^\nu = 1 = -m_\nu m^{*\nu}$ and $m_\nu l^\nu = 0 = n_\nu l^\nu$. The electromagnetic field is then represented by the three complex scalar potentials

$$\phi_0 \equiv F_{\mu\nu} l^\mu m^\nu, \quad \phi_2 \equiv F_{\mu\nu} m^{*\mu} n^\nu,$$

$$\phi_1 \equiv \frac{1}{2} F_{\mu\nu}(l^\mu n^\nu + m^{*\mu} m^\nu), \tag{1}$$

where $F_{\mu\nu}$ is the EM field-strength tensor. In a similar manner the Weyl and Ricci tensors are expressed in terms of complex scalars, and the Einstein-Maxwell equations are expressed in terms of these scalar potentials. See Chandrasekhar (1979) for a self-contained treatment.

Teukolsky (1973) showed that, for a fixed Kerr geometry in Boyer-Lindquist coordinates (t, r, θ, ϕ), ϕ_0 and ϕ_2 (either of which contains the complete solution to the vacuum Maxwell equations) yield separable solutions

$$\phi_0 = e^{i\omega t} e^{im\phi} {}_1S_{m\omega}(\theta) {}_1R_{m\omega}(r) \tag{2}$$

$$\bar\rho^2 \phi_2 = e^{i\omega t} e^{im\phi} {}_{-1}S_{m\omega}(\theta) {}_{-1}R_{m\omega}(r) \tag{3}$$

where $\bar\rho = r - ia\cos\theta$ and ω is positive. The equations for R and S are

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + a^2\omega^2\cos^2\theta + 2a\omega s\cos\theta - \frac{(m + s\cos\theta)^2}{\sin^2\theta} + E - s^2\right] {}_sS_{m\omega}(\theta; E) = 0 \tag{4}$$

and

$$\left[\Delta^{-s}\frac{d}{dr}\Delta^{s+1}\frac{d}{dr} + \frac{K^2 + 2is(r - M)K}{\Delta} - 4ir\omega s - E - 2am\omega - a^2\omega^2 + s(s+1)\right] {}_sR_{m\omega}(r; E). \tag{5}$$

Here, $K \equiv (r^2 + a^2)\omega + am$, $\Delta \equiv r^2 + a^2 - 2Mr$, M and a are the hole's mass and specific angular momentum, and $E(m, \omega)$ is an angular eigenvalue. The 'spin-weight' s takes the value $s = +1$ if the equations are used for ϕ_0, and

$s=-1$ for ϕ_2. Scalar waves have $s=0$ and gravitational waves have $s=\pm 2$.

Chandrasekhar (1979) has shown how to cast the radial equation into a one-dimensional wave equation with potential barrier of the form

$$\left(\frac{d^2}{dr_*^2}+\omega^2\right)Z=VZ,\quad \frac{d}{dr_*}=\frac{\Delta}{\rho^2}\frac{d}{dr}, \tag{6}$$

where Z is some combination $Z=AR+B\frac{d}{dr_*}R$ (A and B are functions of r and θ given in C79; B=0 for $s=0$), and where

$$\rho^2=r^2+\alpha^2 \quad \text{and} \quad \alpha^2=a^2+(am/\omega). \tag{7}$$

The new r_* variable tends to $+\infty$ as $r\to+\infty$, and tends to $\pm\infty$ as $r\to r_+$ (the outer horizon). The two signs are due to the possible double-valuedness of the $r(r_*)$ relation. For $0<\omega<-am/2Mr_+$ – the 'superradiant interval' (Chandrasekhar 1979), the upper sign applies, and the potential can be written

$$V=\frac{\Delta}{\rho^4}\left[\lambda+\frac{\Delta}{\rho^4}|\alpha|(|\alpha|-4r)+2|\alpha|\frac{r-M}{\rho^2}\right], \tag{8}$$

where $\lambda=E+a^2\omega^2+2am\omega$ is a version of the angular eigenvalue. As $r\to r_+$, $r_*\to+\infty$, $V\to 0$, and the solution tends to $Z\to e^{i\omega r_*}$ if one imposes the correct boundary condition at the horizon (finite-amplitude ingoing waves as observed by infalling observers; see Teukolsky 1973) and gives the wave unit amplitude. The solution may be integrated until it nears the singularity at $r=|\alpha|$, where the differential equation in terms of $x\equiv|\alpha|-r$ approaches

$$x^2\frac{d^2Z}{dx^2}-x\frac{dZ}{dx}=-\frac{3}{4}Z, \tag{9}$$

the solution of which is[1] $Z\to C_1x^{3/2}+C_2x^{1/2}$ Once C_1 and C_2 are determined, the integration may be restarted with r slightly greater than $|\alpha|$, with the solution $Z=iC_1|x|^{3/2}-iC_2|x|^{1/2}$. Note that the Wronksian $W_{r_*}[Z,Z^*]$ changes sign at $r=|\alpha|$.

As $r\to\infty$, $r_*\to\infty$ and $Z\to C_{\rm inc}e^{i\omega r_*}+C_{\rm ref}e^{-i\omega r_*}$. This allows the definition of reflection and transmission coefficients

$$R=|C_{\rm ref}|^2|C_{\rm inc}|^{-2},\quad T=|C_{\rm inc}|^{-2} \tag{10}$$

which, due to the change in sign of the Wronskian, obey $R-T=1$ when ω is in the superradiant interval. The numerical method then entails integrating Z from the boundary condition at $r\to r_+$ out to $r\to\infty$ and finding $C_{\rm inc}$ and $C_{\rm ref}$.

The scalar case has a very similar potential, given by[2]

$$V=\frac{\Delta}{\rho^4}\left[\lambda+\frac{1}{\rho^2}[\Delta+2r(r-M)]-3\frac{r^2\Delta}{\rho^4}\right]. \tag{11}$$

The scalar case can be integrated through the singularity using the same method as employed for $s=1$, or by switching back to the usual function $R(r)=Z(r_*)/\sqrt{|\rho^2|}$ and using eq. 5 to integrate past the singularity.

These equations provide a prescription for computing the degree of superradiant reflection given only the angular eigenvalue E appropriate to the angular geometry and the m and ω values.

3. THE WEDGE GEOMETRY

Van Putten considers the simplified problem of a very narrow equatorial wedge. It is then assumed that the angular function is constant across this wedge: $dS/d\theta=0$ in equation 4. Approximating then $\cos\theta\to 0$ and $\sin\theta\to 1$, eq. 4 trivializes to

$$E\simeq m^2+s^2. \tag{12}$$

This process is somewhat like creating an $l=0, m>0$ mode. These eigenvalues contrast with the usual eigenvalues of the 'spin-weighted spheroidal harmonics' (Teukolsky 1973) which follow from boundary conditions of regularity at $\theta=0$ and $\theta=\pi$ and are given (for small $a\omega$) by (Fackerell & Crossman 1977):

$$E=l(l+1)+2a\omega\frac{s^2m}{l(l+1)}+O[(a\omega)^2] \tag{13}$$

for $s=1$, and for $s=0$ by

$$E=l(l+1)+2a^2\omega^2\left[\frac{m^2-l(l+1)+\frac{1}{2}}{(2l-1)(2l+3)}\right]+O[(a\omega)^4].$$

For $s=0$, the wedge geometry changes the angular eigenvalue for $m=1$ from $E=2+O[(a\omega)^2]$ to $E=1$. The eigenvalue adds to the height of the potential barrier, so the lower eigenvalue in the wedge geometry leads to much *higher* superradiance in the scalar case (see §4 below for numerical results).

An immediate worry arises in the $s=1$ case: the wedge geometry gives $E=1+m^2$, which is *higher* than the minimal $l=|m|=1$ eigenvalues, which are <2 (see eq. 13; $m\omega<0$ is required for superradiance.) This suggests that the result obtained for scalar waves will not generalize to higher spins. But before drawing this conclusion firmly we must pose the problem as clearly as possible for the magnetosonic waves.

Van Putten's model postulates a force-free magnetosphere with a current-carrying torus surrounding a current-free toroidal cavity (see Van Putten 1999, fig. 2). Since the boundaries of the region are defined by magnetic field lines, component of $\vec{B}$ perpendicular to the boundary must vanish there. The force-free condition implies that $\vec{E}\perp\vec{B}$. This still leaves a choice of direction in $\vec{E}$. I shall choose *one polarization state*, in which the components of $\vec{E}$ parallel to the boundary must vanish. The boundary conditions are, then, just like those at the boundary of a perfect conductor. In the wedge geometry, these are that $E_r=E_\phi=B_\theta=0$ near $\theta=\pi/2$, and hold in the rest-frame of the matter in which the $\vec{B}$−field is anchored. They also, it turns out, hold in any frame connected to this

[1] Not noted in Chandrasekhar 1979, this dangerously discards terms of $O(xZ)$, the order of the $x^{3/2}$ solution term. But in the next-order expansion of eq. 6, a surprising cancellation justifies the procedure.

[2] This corrects the equation given in Chandrasekhar 1976.

frame by a boost in the $\hat{\phi}$ direction, and therefore the conditions can be specified in the 'locally non-rotating frame' (LNRF; Bardeen 1972) as long as there are predominantly ϕ-direction bulk motions in the matter.

King (1977) gives the relevant LNRF field components explicitly in terms of the Newman-Penrose potentials ϕ_i, in the Boyer-Lindquist coordinates. Evaluated at $\theta \to \pi/2$, the equations $E_r = E_\phi = B_\theta = 0$ give

$$\Im\left[\phi_2 + \frac{\Delta}{2r^2}\phi_0\right] = 0 \qquad (14)$$

$$\Im\left[\phi_2 - \frac{\Delta}{2r^2}\phi_0\right] - \frac{2^{1/2}(r^2+a^2)}{ar}\Re[\phi_1] = 0 \qquad (15)$$

$$\Im\left[\phi_2 - \frac{\Delta}{2r^2}\phi_0\right] - \frac{2^{1/2}a\Delta}{r(r^2+a^2)}\Re[\phi_1] = 0. \qquad (16)$$

Subtracting the last two implies that

$$\left(\frac{r^2+a^2}{a^2} - \frac{r^2+a^2-2Mr}{r^2+a^2}\right)\Re[\phi_1] = 0 \qquad (17)$$

Since its coefficient is always nonzero, we must have $\Re[\phi_1] = 0$, which then (using the first equation) implies that the imaginary parts of ϕ_0 and ϕ_2 vanish. So the necessary and sufficient conditions for the proper field quantities to vanish at $\theta = \pi/2$ are:

$$\Im[\phi_0] = \Im[\phi_2] = \Re[\phi_1] = 0. \qquad (18)$$

Because we have assumed a time dependence $\propto e^{i\omega t}$, in specifying the value of the real or imaginary part of the complex scalars, we must consider modes with frequencies $\pm\omega$; likewise, with azimuthal mode numbers $\pm m$. The radial eigenfunctions (which, due to separation, do not change when adopting the wedge-geometry) obey

$${}_sR_{m\omega}(r;E)^* = {}_sR_{-m-\omega^*}(r;E^*), \qquad (19)$$

therefore the solution, composed of two modes, of

$$\begin{aligned}\phi_0 = {}& S(\theta){}_1R_{m\omega}(r;E)e^{im\phi}e^{i\omega t} \\ &+ S(\theta){}_1R_{-m-\omega}(r;E)e^{-im\phi}e^{-i\omega t}\end{aligned} \qquad (20)$$

has vanishing imaginary part if $S(\theta)$, ω and E are real. Adopting then $S(\theta) = 1$, the 'trivialized' angular equation will be satisfied, for $E = 1 + m^2$. The angular solutions of ϕ_2 are related to those of ϕ_0 by 11

$$\begin{aligned}{}_{-1}S_{m\omega} \propto {}& (\partial_\theta + m\csc\theta + a\omega\sin\theta) \\ &\times(\partial_\theta + m\csc\theta + a\omega\sin\theta + \cot\theta){}_1S_{m\omega}\end{aligned} \qquad (21)$$

Multiplying this out and evaluating for $\theta \to \pi/2$ yields

$${}_{-1}S_{m\omega} \to (\text{const.}) \times [(m+a\omega)^2 - 1]{}_1S_{m\omega}, \qquad (22)$$

so choosing both angular functions to be constant is consistent. Since the radial solutions to for ϕ_2 also satisfy condition 19, this will lead to a vanishing imaginary part of ϕ_2 as well. It can also be shown (using Chandrasekhar 1979, eq. 7.186 for each mode) that $\Re[\phi_1]$ vanishes, so the full boundary conditions are satisfied by eq. 20

Since the Maxwell equations are linear, we can evolve the two radial solutions $R_{lm\omega}$ and $R_{l-m-\omega}$ independently from $r = r_+$ to $r \to \infty$ to obtain the incoming and outgoing wave amplitudes. The amplitudes so obtained are invariant under $m \to -m, \omega \to -\omega$, therefore the reflection and transmission coefficients so obtained will be just those obtained by considering either mode. This shows that the superradiance of linearly polarized electromagnetic waves in a thin wedge between perfect conductors can be calculated using of the equations outlined in §2. Note that a particular polarization has been chosen (the other polarization state would have boundary conditions which depend on the background magnetic field configuration), and magnetosonic wave have been shown to coincide with linearly polarized EM waves only in the geometrical-optics limits. It is therefore possible that magnetosonic superradiance will be different, but this difference is unlikely to be large.

4. RESULTS AND DISCUSSION

I have calculated the degree of superradiant reflection for electromagnetic waves using both the usual eigenvalues as tabulated by Press & Teukolsky (1974), and using the eigenvalues for the 'wedge approximation' of $E = m^2 + s^2$. These are shown, in fig. 1, with $l = m = 1$ and for various values of a, as functions of ω. The maximum superradiance in the 'usual' electromagnetic case is $\approx 4.4\%$, and this falls to $\sim 1\%$ in the wedge approximation. I have also computed the scalar wave superradiance, also shown in fig. 1. Van Putten employed the WKB approximation to estimate scalar superradiance, but the potential varies over a scale comparable to the mode wavelength. The $s = 0$ results show that Van Putten's use of the WKB approximation is not very accurate, and that in fact the scalar-wave superradiance increases in the wedge geometry even more than he predicts, rising to a maximum value of $\sim 7\%$, from a maximum of $\sim 0.3\%$ in the full geometry.

The somewhat counter-intuitive result that superradiance decreases in the wedge geometry for $s = 1$ while increasing for $s = 0$ can be understood using the following heuristic argument. The angular eigenvalue E links the angular and radial parts of the wave equation, effectively adding a term to the potential representing the angular momentum barrier; this situation is familiar from quantum mechanics, where (as in the $\omega = 0$ case here), $l(l+1)$ gives the total angular momentum $\mathcal{L}^2$ associated with the eigenfunction labeled by (l, m). Also familiar from quantum mechanics,

$$\mathcal{L}^2 = \langle\vec{L}^2\rangle = \langle L_x^2\rangle + \langle L_y^2\rangle + \langle L_z^2\rangle \geq \langle L_z^2\rangle = m^2,$$

i.e. m^2 is a lower limit to the total angular momentum (resulting from the azimuthal variation), regardless of the value of l (which is, of course, always $\geq m$ in the full-sphere case). Generalizing this to $s \geq 0$, it is possible to construct an operator K_r, analogous to L_z but with eigenvalue s, representing the angular momentum (helicity) about the radial direction, rather than about the $\hat{z}$ direction (Goldberg et al. 1967; Campbell 1972). For $\theta = \pi/2$, the radial direction lies in the $\hat{x} - \hat{y}$ plane, and we can write

$$\mathcal{L}^2 = \langle L_x^2\rangle + \langle L_y^2\rangle + \langle L_z^2\rangle = L_z^2 + K_r^2 = m^2 + s^2 \qquad (23)$$

These equations imply that the total angular momentum has a value $m^2 + s^2$ when the wave is confined to $\theta = \pi/2$, in agreement with the 'trivialized' angular equation 12. That is, the helicity of the $s > 0$ wave provides an extra component of the angular momentum barrier in the wedge which is not present in the scalar case.

The phenomenon of superradiant scattering from a Kerr hole has been understood for thirty years, but has yet to find astrophysical applications because the degree of amplification for electromagnetic waves tends to be small; creation of an instability requires a very efficient 'mirror' with reflectivity of $\gtrsim 95\%$. Van Putten only assumes $0.5 - 5\%$ superradiance in calculating timescales in his model, but his interesting analysis of the thin equatorial wedge suggested that the wedge geometry might greatly enhance superradiance, making its astrophysical importance very plausible. Unfortunately the present, more detailed, calculations do not bear out this idea. The cavity in which the magnetosonic waves are confined must be $\sim 99\%$ dissipation-free to create an instability; the assumption that such cavities can form in a natural setting requires justification. Moreover, energy leaking to higher-m or higher-ω modes will be even more weakly amplified. A remaining possibility for the importance of superradiance in a less idealized setting remains, however. If the wave is reflected (or perhaps trapped) close to the resonance radius $r = |\alpha|$ (rather than at infinity), the outer part of the potential barrier could be avoided and superradiance increased. Whether this increase might be sufficient to be astrophysically relevant requires further analysis.

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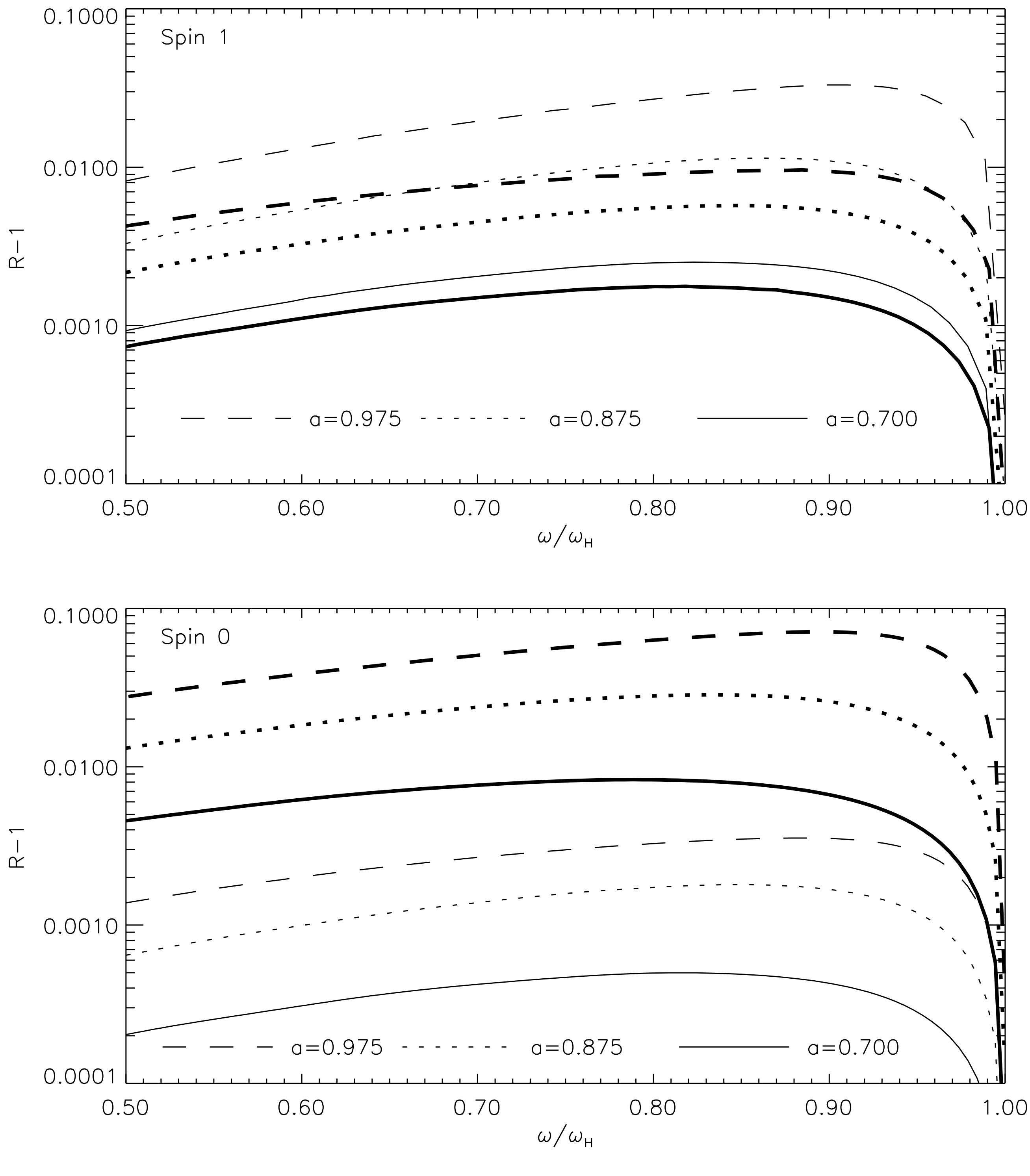

FIG. 1.— **Top:** Amplification $R-1$ for spin-1 (electromagnetic/magnetosonic) waves for various a, at frequencies normalized to $\omega_H \equiv -am/2Mr_+$. Thin lines are with 'usual' eigenvalues; thick lines use the 'wedge approximation' $E = m^2 + s^2$. **Bottom:** Same, for spin-0 (scalar) waves.